COVERING THE CIRCLE WITH RANDOM ARCS

BY

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ABSTRACT

Arcs of lengths l_n, $0 < l_{n+1} \leqq l_n < 1$, $n = 1, 2, \ldots$, are thrown independently and uniformly on a circumference C of unit length. The union of the arcs covers C with probability one if and only if

(1) $$\sum_{n=1}^{\infty} n^{-2} \exp(l_1 + \cdots + l_n) = \infty .$$

1. Introduction

We say that a given sequence $\{l_n\}$ *covers* if almost surely (a.s.) every point of C belongs to some arc. If $\sum l_n = \infty$, each *fixed* point $x \in C$ is a.s. covered by some arc because

$$P_r(x \text{ is not covered}) = \prod_{n=1}^{\infty} (1 - l_n) = 0.$$

Dvoretzky [2] gave the first examples of sequences $\{l_n\}$ with $\sum l_n = \infty$ which do not cover and posed the problem settled here of finding necessary and sufficient conditions that $\{l_n\}$ covers.

Billard [1], and later Kahane [4] in a very elegant way, proved: (a)

(2) $$\limsup_{n \to \infty} 1/n \exp(l_1 + \cdots + l_n) = \infty$$

is sufficient for covering, and (b)

(3) $$\sum_{n=1}^{\infty} l_n^2 \exp(l_1 + \cdots + l_n) = \infty$$

is necessary for covering. Orey [7], using topological as well as probabilistic methods to study the number of components of the union of the first n arcs, improved (a) by showing that:

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(2′) $$\limsup_{n\to\infty} 1/n \exp(l_1 + \cdots + l_n) > 0$$

is sufficient for covering. The case $l_n = 1/(n+1)$ which had remained unsettled until Orey's work was independently shown to be a case of covering by Mandelbrot [5], [6]. Mandelbrot considered an interesting related problem where the lengths l_n are also random and succeeded in applying the results to settle the above case in particular.*

We obtain that (1) is a necessary and sufficient condition for covering and in Sec. 7 show that the previous results follow directly from (1). If $0 < l_n < 1$ but $\{l_n\}$ is not monotonically decreasing then covering takes place if and only if (1) applies to the sequence $\{l_n\}$ rearranged in decreasing order; if l_n cannot be rearranged in decreasing order, l_n does not tend to zero and covering takes place trivially because an infinite subsequence of $\{l_n\}$ can be found bounded away from zero (and (1) can then be used.) Thus (1) settles the problem completely.

In Sec. 7, we show that whenever C is covered, it is covered infinitely often almost surely and also give some examples and remarks. Finally in Sec 8, we obtain some new results on the distribution of the time to cover C by arcs of equal lengths.

We use a new method to prove that (1) is sufficient for covering based on conditioning with respect to the first uncovered point in an orientation of C.

2. The basic lemma

Let n arcs (open intervals) $I_1, I_2, \cdots, I_n$ of lengths $l_1, l_2, \cdots, l_n$ be obtained by choosing their centers independently and uniformly distributed on C. Denote $U_n = \bigcup_{j=1}^n I_j$.

Suppose $A \subset C$ is a fixed finite set of points and t and θ are points of C not in A.

Property π. We say that t, θ, A have property π if any arc of length max $(l_1, \cdots, l_n)$ which contains t and intersects A must also contain θ. We picture θ as lying between t and A with t close to A.

Lemma 1. *If t, θ, A have property π then*

(4) $$P(A \subset U_n \mid \theta \notin U_n) \leqq P(A \subset U_n \mid \theta \notin U_n, t \notin U_n).$$

We remark that (4) is intuitive since any arc in U_n which covers t cannot touch A

* The inequalities (16) and (26) below yield much more: In particular it follows for $l_n = 1/(n+1)$ that p_n = the pro bability that the first n arcs fail to cover satisfies $0 < c_1 < p_n \log(n+1) < c_2 < \infty$.

under the condition $\theta \notin U_n$. To prove (4), let J_1 and K_1 be arcs of length l_1 obtained as follows: Let $I_1^1, I_1^2, \cdots$ be arcs of length l_1 each chosen independently and uniformly on C. Let J_1 be the first arc I_1^j for which $\theta \notin I_1^j$, and let K_1 be the first arc I_1^k for which $\theta \notin I_1^k$ and $t \notin I_1^k$, noting that $k \geqq j$. Let $J_2, \cdots, J_n$ and $K_2, \cdots, K_n$ be defined similarly. Since the distribution of $J_1, \cdots, J_n$ is the same as the conditional distribution of $I_1, \cdots, I_n$ under the condition $\theta \notin U_n$,

$$P(A \subset U_n \mid \theta \notin U_n) = P\left(A \subset \bigcup_{i=1}^{n} J_i\right) \tag{5}$$

Since the distribution of $K_1, \cdots, K_n$ is the same as the conditional distribution of $I_1, \cdots, I_n$ under the condition $\theta \notin U_n$ and $t \notin U_n$,

$$P(A \subset U_n \mid \theta \notin U_n,\, t \notin U_n) = P\left(A \subset \bigcup_{i=1}^{n} K_i\right). \tag{6}$$

But if $t \in J_i$ then $J_i \cap A = \varnothing$ because of property π, while if $t \notin J_i$ then $J_i = K_i$. Thus in any case $J_i \cap A \subset K_i \cap A$, $i = 1, \cdots, n$ and so

$$P\left(A \subset \bigcup_{i=1}^{n} J_i\right) \leqq P\left(A \subset \bigcup_{i=1}^{n} K_i\right). \tag{7}$$

(4) is immediate from (5), (6) and (7).

Finally using Bayes' rule we can rewrite (4) as

$$P(t \notin U_n \mid \theta \notin U_n) \leqq P(t \notin U_n \mid A \subset U_n, \theta \notin U_n). \tag{8}$$

3. Covering; sufficiency

Coordinatize $C = \{0 \leqq t < 1\}$ in counterclockwise orientation and let $I_i = (\theta_i', \theta_i)$ where θ_i' and θ_i are the coordinates of the endpoints of I_i with θ_i' clockwise from θ_i along points of I_i. Let $\omega = (\theta_1, \cdots, \theta_n)$ and $U_n(\omega) = U_n = \bigcup_{j=1}^{n} I_j$. Set

$$\chi(t,\omega) = \begin{cases} 1 & \text{if } t \notin U_n(\omega), \\ 0 & \text{if } t \in U_n(\omega) \end{cases} \qquad 0 \leqq t < 1. \tag{9}$$

The Lebesgue measure of the *uncovered part of* the arc (a,b), $0 \leqq a < b < 1$ is given by

$$m(a,b,\omega) = \int_a^b \chi(t,\omega)\,dt. \tag{10}$$

We have with $m(a,b) = m(a,b,\omega)$,

$$E\,m(a,b) = \int_a^b P(t \notin U_n)\,dt = (b-a)P(0 \notin U_n). \tag{11}$$

Choose and fix $\varepsilon > 0$ for which

$$\varepsilon < 1 - l_i, \qquad i = 1,\cdots,n. \tag{12}$$

Heuristically, the idea of the covering method is as follows: Let $\tau(\omega)$ be the first uncovered point of $[0,\varepsilon)$ if there is one. Then

$$\begin{aligned} Em(0,\varepsilon) &= \int_0^{\varepsilon} dP(\tau \leqq \theta)E[m(\theta,\varepsilon)\,|\,\tau = \theta] \\ &= \int_0^{\varepsilon} dP(\tau \leqq \theta)E[m(\theta,\varepsilon)\,|\,[0,\theta) \subset U_n, \theta \notin U_n] \\ &\geqq \int_0^{\varepsilon/2} dP(\tau \leqq \theta)E[m(\theta,\theta+\varepsilon/2)\,|\,[0,\theta)\subset U_n, \theta\notin U_n] \\ &\geqq \int_0^{\varepsilon/2} dP(\tau \leqq \theta)E[m(\theta,\theta+\varepsilon/2)\,|\,\theta\notin U_n] \end{aligned} \tag{13}$$

by using (8) with $A = [0,\theta)$ integrated over $t \in (\theta, \theta + \varepsilon/2)$, noting that π holds by (12). The integrand in the last term is clearly not a function of θ by rotational symmetry so we may put $\theta = 0$. This gives

$$Em(0,\varepsilon) \geqq P(\tau \leqq \varepsilon/2)E[m(0,\varepsilon/2)\,|\,0 \notin U_n]. \tag{14}$$

Since from (10) and (11) we have

$$\begin{aligned} &Em(0,\varepsilon) = \varepsilon P(0\notin U_n) \\ &E[m(0,\varepsilon/2)\,|\,0\notin U_n] = \int_0^{\varepsilon/2} P(t\notin U_n\,|\,0\notin U_n)\,dt \end{aligned} \tag{15}$$

and since $\tau \leqq \varepsilon/2$ if and only if $[0\ \varepsilon/2]$ is not covered we get the useful inequality

$$P([0,\varepsilon/2)\not\subset U_n) \leqq \varepsilon P(0\notin U_n)\Big/\int_0^{\varepsilon/2} P(t\notin U_n\,|\,0\notin U_n)\,dt. \tag{16}$$

The above argument while convincing is not a proof because the second inequality in (13) is obtained from (8) in a case where A is not finite and $\theta \in$ closure of A which is not permissible since then the right hand side of (8) involves conditioning with respect to a set of probability zero. Nevertheless (16) is true and the object of the rest of this section is a precise proof of (16). An earlier unpublished but widely circulated version of this paper gave a much more complicated proof of (16) than the one below.

Fix k and let $\tau_k = j/k$ if $0,\ 1/k,\cdots,(j-1)/k \in U_n$ but $j/k \notin U_n$, $j = 0\ 1,2,\cdots$. Then

$$
\begin{aligned}
E\,m(0,\varepsilon) &= \sum_{0\leqq j\leqq k\varepsilon} P(\tau_k = j/k)E(m(0,\varepsilon)\,|\,\tau_k = j/k) \\
&\geqq \sum_{0\leqq j\leqq k\varepsilon/2} P(\tau_k = j/k)E(m(j/k, j/k+\varepsilon/2)\,|\,\tau_k = j/k) \qquad (17)\\
&= \sum_{0\leqq j\leqq k\varepsilon/2} P(\tau_k = j/k)\int_{j/k}^{j/k+\varepsilon/2} P(t\notin U_n\,|\,\tau_k = j/k)\,dt
\end{aligned}
$$

where we have used (10) in the last line. The event $\{\tau_k = j/k\}$ is the same as $\{A\subset U_n,\ \theta\notin U_n\}$ where $A = \{0, 1/k, \cdots, (j-1)/k\}$, and $\theta = j/k$. Applying (8) to the integrand in the last line of (17) we obtain with $t = (j/k)+s$,

$$
(18)\qquad E\,m(0,\varepsilon)\geqq \sum_{0\leqq j\leqq k\varepsilon/2} P(\tau_k = j/k)\int_0^{\varepsilon/2} P((j/k)+s\notin U_n\,|\,j/k\notin U_n)\,ds.
$$

By rotational symmetry we obtain

$$
(19)\qquad \int_0^{\varepsilon/2} P((j/k)+s\notin U_n\,|\,j/k\notin U_n)\,ds = \int_0^{\varepsilon/2} P(t\notin U_n\,|\,0\notin U_n)\,dt.
$$

Using (15) (18) and (19) we obtain

$P(\text{for some } j,\ 0\leqq j\leqq k\,\varepsilon/2,\ j/k\notin U_n) =$

$$
(20)\qquad \sum_{0\leqq j\leqq k\varepsilon/2} P(\tau_k = j/k)\leqq \varepsilon P(0\notin U_n)\Big/\int_0^{\varepsilon/2} P(t\notin U_n\,|\,0\notin U_n)\,dt.
$$

As $k\to\infty$ through powers of 2 say, the event on the left in (20) increases to $\{Q[0,\varepsilon/2)\subset U_n\}$ where $Q[0,\varepsilon/2)$ denotes the set of dyadic rationals in $[0,\varepsilon/2)$. Since U_n is a finite set of intervals the latter event differs at most by a null set from $\{[0,\varepsilon/2)\not\subset U_n\}$. Hence (16) is proved.

4. Covering; necessity

Here we essentially follow the method of Billard and Kahane [4].

Define ω, $I_1,\cdots,I_n, U_n = \bigcup_{j=1}^n I_j$, $m(0,\varepsilon) = m(0,\varepsilon,\omega)$ as the measure of the uncovered part of $[0,\varepsilon)$ just as before and set

$$
(21)\qquad \phi(\omega) = \begin{cases} 1 & [0,\varepsilon)\not\subset U_n(\omega) \\ 0 & [0,\varepsilon)\subset U_n(\omega). \end{cases}
$$

Since either $\phi(\omega) = 1$ or $m(0,\varepsilon,\omega) = 0$ we have

$$
(22)\qquad m(0,\varepsilon,\omega) = \phi(\omega)m(0,\varepsilon,\omega).
$$

Applying Schwarz's inequality,

$$
(23)\qquad (E\,m(0,\varepsilon))^2 \leqq E\,\phi^2 E\,m^2(0,\varepsilon)
$$

Since $\phi^2 = \phi$, from (11) we obtain

$$P([0,\varepsilon) \not\subset U_n) \geqq \varepsilon^2 P^2(0 \notin U_n)/E\,m^2(0,\varepsilon) \tag{24}$$

But from (10) and rotational symmetry,

$$\begin{aligned} E\,m^2(0,\varepsilon) &= E\int_0^\varepsilon \int_0^\varepsilon \chi(s,\omega)\chi(t,\omega)\,ds\,dt \\ &= \int_0^\varepsilon ds \int_0^\varepsilon dt\,P(t \notin U_n, s \notin U_n) \\ &= 2\int_0^\varepsilon ds \int_0^s dt\,P(t \notin U_n, s \notin U_n) \\ &= 2\int_0^\varepsilon ds \int_0^s dt\,P(t - s \notin U_n, 0 \notin U_n) \\ &= 2\int_0^\varepsilon ds \int_0^s dt\,P(t \notin U_n, 0 \notin U_n) \\ &= 2\int_0^\varepsilon dt \int_t^\varepsilon ds\,P(t \notin U_n, 0 \notin U_n) \\ &= 2\int_0^\varepsilon (\varepsilon - t)P(t \notin U_n, 0 \notin U_n)\,dt \\ &\leqq 2\varepsilon \int_0^\varepsilon P(t \notin U_n, 0 \notin U_n)\,dt. \end{aligned} \tag{25}$$

From (24) and (25), replacing ε by $\varepsilon/2$,

$$P([0,\varepsilon/2) \not\subset U_n) \geqq (\tfrac{1}{4})\varepsilon P(0 \notin U_n) \Big/ \int_0^{\varepsilon/2} P(t \notin U_n \,|\, 0 \notin U_n)\,dt. \tag{26}$$

Thus (16) is valid with the inequality sign reversed except for a factor $\frac{1}{4}$.

5. Infinitely many arcs, $n \to \infty$

Denote $U = \bigcup_{n=1}^{\infty} I_n$. By the Heine Borel theorem,

$$\{[0,\varepsilon] \not\subset U\} = \bigcap_{n=1}^{\infty} \{[0,\varepsilon] \not\subset U_n\}, \tag{27}$$

and since the events $\{[0,\varepsilon] \subset U_n\}$ decrease

$$P([0,\varepsilon] \not\subset U) = \lim_{n\to\infty} P([0,\varepsilon] \not\subset U_n). \tag{28}$$

If $1 > l_1 \geqq l_2 \cdots$ and $2\varepsilon < 1 - l_1$ then (12) holds for all n and so (16) and (26) apply to the terms on the right side of (28). It follows from (16) that

$$P([0,\varepsilon] \not\subset U) \leqq 2\varepsilon \liminf_{n\to\infty} \frac{P(0 \notin U_n)}{\int_0^\varepsilon P(t \notin U_n \mid 0 \notin U_n)\,dt} \tag{29}$$

and from (26) that

$$P([0,\varepsilon] \not\subset U) \geqq \varepsilon/2 \limsup_{n\to\infty} \frac{P(0 \notin U_n)}{\int_0^\varepsilon P(t \notin U_n \mid 0 \notin U_n)\,dt}. \tag{30}$$

LEMMA 2. *C is covered a.s. if and only if*

$$\limsup_{n\to\infty} \int_0^\varepsilon \frac{P(t \notin U_n, 0 \notin U_n)}{[P(0 \notin U_n)]^2}\,dt = \infty \tag{31}$$

for all $\varepsilon > 0$ (equivalently, for some $\varepsilon > 0$).

PROOF. If C is covered a.s. then $[0,\varepsilon]$ is covered a.s. for all $\varepsilon > 0$ and so the left side of (30) is zero. Thus (31) holds for all $\varepsilon > 0$ since the term under the lim sup in (30) is the reciprocal of the one in (31). Conversely, if (31) holds for some $\varepsilon_0 > 0$ then the right side of (29) is zero and so $[0,\varepsilon_0]$ is covered with probability one. But so then by rotational symmetry are $[\varepsilon_0, 2\varepsilon_0]$, $[2\varepsilon_0, 3\varepsilon_0], \cdots$ covered a.s. and hence C is also covered a.s. This proves the lemma.

REMARK. By (29) and (30) the lim sup in (31) could equivalently be replaced by lim inf.

6. The condition

As justified in the introduction, we assume from now on that $\{l_n\}$ has been rearranged in nonincreasing order. It remains to determine for which $\{l_n\}$, (31) holds. By independence of $I_1, \cdots, I_n$,

$$P(0 \notin U_n) = P\left(\bigcap_{j=1}^n \{0 \notin I_j\}\right) = \prod_{j=1}^n P(0 \notin I_j) = \prod_{j=1}^n (1 - l_j), \tag{32}$$

and

$$\begin{aligned} P(t \notin U_n, 0 \notin U_n) &= P\left(\bigcap_{j=1}^n \{t \notin I_j, 0 \notin I_j\}\right) \\ &= \prod_{j=1}^n P(t \notin I_j, 0 \notin I_j) \end{aligned} \tag{33}$$

If $0 \leqq t \leqq \frac{1}{2}$ it is easy to see that

$$\begin{aligned} P(t \notin I_j, 0 \notin I_j) &= P(t \notin I_j) - P(t \notin I_j, 0 \in I_j) \\ &= 1 - l_j - \min(l_j, t). \end{aligned} \tag{34}$$

Thus the necessary and sufficient condition for covering, (31), becomes

$$\limsup_{n\to\infty} \int_0^{\varepsilon} \prod_{j=1}^{n} [(1 - l_j - \min(l_j,t))/(1-l_j)^2]\,dt = \infty. \tag{35}$$

for some or all $\varepsilon > 0$.

The remainder of the proof is of course to show that (35) is equivalent to (1), which is an immediate consequence of the following two lemmas. Again $\{l_n\}$ is assumed nonincreasing.

LEMMA 3. *If* $\sum l_n^2 < \infty$ *then* (35) *is equivalent to* (1).

LEMMA 4. *If* $\sum l_n^2 = \infty$ *then* (35) *and* (1) *both hold.*

To prove Lemma 3, choose K so that $l_K < \frac{1}{2}$ and set $\varepsilon = l_K$. Recalling that $l_K \geqq l_{K+1} \geqq \cdots \geqq l_n$, for $n \geqq K$, the integral in (35) for $\varepsilon = l_K$ breaks up for $n \geqq K$ into the sum of integrals over the intervals $(0, l_n), (l_n, l_{n-1}), \cdots, (l_{K+1}, l_K)$. Noting that $\min(l_j, t) = l_j$ for $l_{k+1} < t < l_k$ and $j > k$; $\min(l_j,t) = t$ for $l_{k+1} < t < l_k$ and $j \leqq k$, the integral in (35) becomes

$$\begin{aligned} &\int_0^{l_k} \prod_{j=1}^{n} [(1 - l_j - \min(l_j,t))/(1-l_j)^2]\,dt \\ &= \sum_{k=K}^{n-1} \int_{l_{k+1}}^{l_k} \prod_{j=1}^{k} [(1 - l_j - t)/(1-l_j)^2] \prod_{j=k+1}^{n} [1 - (l_j/(1-l_j))^2]\,dt \\ &\quad + \int_0^{l_n} \prod_{j=1}^{n} [(1 - l_j - t)/(1-l_j)^2]\,dt. \end{aligned} \tag{36}$$

Since l_n is square summable, the product

$$\prod_{j=k+1}^{n} [1 - (l_j/(1-l_j))^2], \tag{37}$$

is uniformly bounded away from zero and infinity in the range $0 \leqq k \leqq n < \infty$. Also, the product

$$\prod_{j=1}^{k} \frac{1 - l_j - t}{(1-l_j)^2} = \prod_{j=1}^{k} (1 + l_j - t + O(l_j^2)) \tag{38}$$

may be replaced in (36) by

$$\exp[l_1 + \cdots + l_k - kt] \tag{39}$$

since the ratio of (38) to (39) is for $0 \leqq t \leqq l_k$,

$$\prod_{j=1}^{k} (1 + O(l_j^2)) \tag{40}$$

which is bounded away from zero and infinity since $\sum l_j{}^2 < \infty$. Thus (35) becomes

$$\begin{aligned} \infty = \limsup_{n\to\infty} \Bigg[&\sum_{k=K}^{n-1} \int_{l_{k+1}}^{l_k} \exp(l_1 + \cdots + l_k - kt)\,dt \\ &+ \int_0^{l_n} \exp(l_1 + \cdots + l_n - nt)\,dt \Bigg]. \end{aligned} \tag{41}$$

Performing the integrations, the nth term in (41) becomes with $s_k = l_1 + \cdots + l_k$,

$$\begin{aligned} &\sum_{k=K}^{n-1} [\exp(s_{k+1} - (k+1)l_{k+1}) - \exp(s_k - kl_k)]/k \\ &\qquad + 1/n \exp s_n - 1/n \exp(s_n - nl_n). \end{aligned} \tag{42}$$

Summing by parts, (42) becomes

$$\begin{aligned} &1/n \exp s_n + \sum_{k=K+1}^{n} [\exp(s_k - kl_k)]/k(k-1) \\ &\qquad - \exp(s_K - Kl_K)/K. \end{aligned} \tag{43}$$

Thus if $\sum l_n^2 < \infty$, (35) holds if and only if at least one of

$$\limsup_{n\to\infty} 1/n \exp(l_1 + \cdots + l_n) = \infty \tag{44}$$

or

$$\sum_{n=1}^{\infty} 1/n^2 \exp(l_1 + \cdots + l_n - nl_n) = \infty, \tag{45}$$

holds. Lemma 3 now follows easily from the next two lemmas, since if $\sum l_n^2 < \infty$, l_n decreases to zero and so (35) $\Leftrightarrow$ (44) or (45) (as shown above) $\Leftrightarrow$ (45) (by lemma 5) $\Leftrightarrow$ (1) by lemma 6).

LEMMA 5. *If l_n decreses to zero,* (44) *implies* (45).

LEMMA 6. *If l_n decreases to zero,* (45) *is equivalent to* (1).

PROOF OF LEMMA 5. Define for all $n \geqq 1$,

$$\xi_n = l_1 + \cdots + l_n - nl_n \tag{46}$$

and observe that ξ_n increases with n since

$$\xi_{n+1} - \xi_n = n(l_n - l_{n+1}). \tag{47}$$

Dividing by n and summing over $n \geqq k$ we see that

$$l_k = \sum_{n \geqq k} (\xi_{n+1} - \xi_n)/n, \qquad k \geqq 1. \tag{48}$$

Summing by parts we obtain for $k \geqq 1$,

$$l_1 + \cdots + l_k = k \sum_{n>k} \xi_n[1/(n-1) - 1/n]. \tag{49}$$

Suppose now that (44) holds but (45) fails and the sum in (45) takes the value $\exp M < \infty$. Since

$$\sum_{n \geqq k} 1/n^2 \geqq \sum_{n \geqq k} [1/n - 1/(n+1)] = 1/k, \tag{50}$$

we have since ξ_n increases,

$$\exp M \geqq \sum_{n \geqq k} (1/n^2) \exp \xi_n \geqq \sum_{n \geqq k} 1/n^2 \exp \xi_k \geqq 1/k \exp \xi_k. \tag{51}$$

Thus $\xi_k \leqq M + \log k$, $k \geqq 1$ and from (49) we obtain

$$\begin{aligned} l_1 + \cdots + l_k &\leqq M + k \sum_{n>k} \log n [1/(n-1) - 1/n] \\ &= M + \log k + 1 + k \sum_{n>k} (\log(n+1) - \log n)/n \\ &\leqq M + 1 + \log k + k \sum_{n>k} 1/n^2 \\ &\leqq M + 2 + \log k, \end{aligned} \tag{52}$$

where we have used $\log n + 1 - \log n = \log(1 + (1/n)) \leqq 1/n$. Thus, for all k

$$1/k \exp(l_1 + \cdots + l_k) \leqq \exp(M+2). \tag{53}$$

and so if (45) fails so does (44). Lemma 5 is proved.

Proof of Lemma 6. The implication (45) $\Rightarrow$ (1) is trivial since $nl_n \geqq 0$. To prove (1) $\Rightarrow$ (45), we note that if (45) fails,

$$\sum_{n=1}^{\infty} \phi_n < \infty \tag{54}$$

where

$$\phi_n = (1/n^2) \exp \xi_n \tag{55}$$

and ξ_n is again given by (46). Since the exponential function is convex and

$$k \sum_{n>k} [1/(n-1) - 1/n] = 1, \tag{56}$$

we obtain

$$\exp\left(k \sum_{n>k} \log \phi_n [1/(n-1) - 1/n]\right) \leqq k \sum_{n>k} \phi_n [1/(n-1) - 1/n]. \tag{57}$$

Using (49), we obtain from (57) since $\xi_n = \log(n^2\phi_n)$

$$\begin{aligned} \exp(l_1 + \cdots + l_k) &= \exp\left\{k \sum_{n>k} [\log(n^2\phi_n)](1/(n-1) - 1/n)\right\} \\ &\leqq \left[\exp\left\{2k \sum_{n>k} \log n(1/(n-1) - 1/n)\right\}\right] \left[k \sum_{n>k} \phi_n(1/n-1) - 1/n)\right]. \end{aligned} \tag{58}$$

As in (52) we obtain that the first term in square brackets is bounded by $\exp\{2(2 + \log k)\}$. Thus,

$$\exp(l_1 + \cdots + l_k) \leqq (\exp 4)k^2 k \sum_{n>k} \phi_n(1/(n-1) - 1/n). \tag{59}$$

Dividing by k^2 and summing, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} 1/k^2 \exp(l_1 + \cdots + l_k) &\leqq (\exp 4) \sum_{k=1}^{\infty} \sum_{n>k} k\phi_n(1/(n-1) - 1/n) \\ &= (\exp 4) \sum_{n=1}^{\infty} \left(\sum_{k<n} k\right) \phi_n(1/(n-1) - 1/n) = \tfrac{1}{2}(\exp 4) \sum_{n=1}^{\infty} \phi_n < \infty \end{aligned} \tag{60}$$

because of (54). Thus (1) fails if (45) fails and Lemma 6 is proved. As observed above this completes the proof of Lemma 3.

Proof of Lemma 4. It seems difficult to prove directly that $\sum l_n^2 = \infty$ implies (35) holds. We proved that (35) holds if and only if C is covered hence Lemma 4 follows immediately from the next three lemmas.

Lemma 7. *If* $\sum_{n=1}^{\infty} l_2^2 = \infty$ *then* (44) *holds.*

Lemma 8. *If* (44) *holds then* C *is covered.*

Lemma 9. *If* (44) *holds then* (1) *holds.*

Proof of Lemma 7. If (44) fails there is an $M < \infty$ for which

$$\sum_{i=1}^{n} l_i \leqq \log n + M, \qquad n \geqq 1. \tag{61}$$

Since l_n are nonincreasing,

$$l_n \leqq 1/n \sum_{i=1}^{n} l_i \leqq \frac{(\log n + M)}{n}, \qquad n \geqq 1 \tag{62}$$

and so $\sum l_n^2 < \infty$. Thus Lemma 7 is proved.

PROOF OF LEMMA 8. This is proved in [4, p. 89]. We give the short proof for completeness. Define as before

$$U_n = \bigcup_{j=1}^{n} I_j. \tag{63}$$

If $C \not\subset U_n$ there is at least one I_k, $k = 1,\cdots,n$ whose counterclockwise endpoint θ_k is not covered by I_j for $j = 1,\cdots,n, j \neq k$. Thus

$$P(C \not\subset U_n) \leqq \sum_{k=1}^{n} P\left(\bigcap_{\substack{j=1 \\ j\neq k}}^{n} \{\theta_k \notin I_j\}\right). \tag{64}$$

By independence the kth term of the sum is

$$\prod_{\substack{j=1 \\ j\neq k}}^{n} P(\theta_k \notin I_j) = \left[\prod_{j=1}^{n} (1 - l_j)\right]/(1 - l_k). \tag{65}$$

From (65) and (66),

$$\begin{aligned} P(C \not\subset U_n) &\leqq \left[\sum_{k=1}^{n} (1 - l_k)^{-1}\right] \prod_{j=1}^{n} (1 - l_j) \\ &\leqq n(1 - l_1)^{-1} \prod_{j=1}^{n} (1 - l_j) \\ &\leqq n(1 - l_1)^{-1}\exp[-(l_1 + \cdots + l_n)]. \end{aligned} \tag{66}$$

Using (28) with $\varepsilon = 1$, and letting $n \to \infty$, we obtain

$$P\{C \not\subset U\} = \lim_{n\to\infty} P\{C \not\subset U_n\} = 0, \tag{67}$$

because the lim inf of the last term in (66) is zero if (44) holds. This proves Lemma 8.

PROOF OF LEMMA 9. If (44) holds and $l_n \to 0$ then (1) holds because of lemmas 5 and 6. If l_n does not tend to zero then there is a $\delta > 0$ for which $l_n > \delta$ for all n and (1) holds trivially. Thus Lemma 9 is proved.

Since Lemmas 3 and 4 are proved, (1) has been shown to be necessary and sufficient for covering.

7. Examples and remarks

REMARK 1. *If C is covered a.s., then C is covered infinitely often a.s.*

PROOF. Denote

(68) $$V_n = \bigcup_{j=n+1}^{\infty} I_j, \qquad n = 1,2,\cdots.$$

Note that the event that some point of C belongs to only finitely many intervals I_j can be written as

(69) $$\bigcup_{n=1}^{\infty} \{C \not\subset V_n\}.$$

Fix n and choose $\varepsilon > 0$ so that

(70) $$\max(l_1,\cdots,l_n) + \varepsilon < 1$$

and let I be the interval with clockwise endpoint at zero and of length $\max(l_1,\cdots,l_n) + \varepsilon$. Since U_n and V_n are independent,

(71) $$\begin{aligned} P(C - I \not\subset V_n)P(U_n \subset I) &= P(C - I \not\subset V_n,\, U_n \subset I) \\ &\leqq P(C \not\subset U) = 0. \end{aligned}$$

Since $P(U_n \subset I) > 0$, we see that

(72) $$P(C - I \not\subset V_n) = 0.$$

Since $C - I$ is an interval and since C is a union of translates of $C - I$ it follows from (72) that

(73) $$P(C \not\subset V_n) = 0.$$

Thus the union in (69) also has probability zero and the remark follows.

We have followed [4] in taking the intervals $I_1, I_2, \cdots$ to be open. This simplifies the uses of the Heine-Borel theorem in (27) to show that events such as $\{C \subset U\}$ are actually measurable. However, it is easy to show that taking the intervals closed or half-open changes none of the results.

Since (2′) is known [7] to be sufficient for covering, we must have (2′)$\Rightarrow$(1). A direct proof that (2′)$\Rightarrow$(1) can be given as follows. Suppose that (2′) holds but (45) does not. Then for any $\varepsilon > 0$ there is an N for which for $k \geqq N$,

(74) $$\sum_{n=k}^{\infty} 1/n^2 \exp \xi_n < \varepsilon$$

where ξ_n are defined by (46). From (51), $\exp \xi_k \leqq k\varepsilon$ for $k \geqq N$, and from (49) and (52), with $M = \log \varepsilon$,

(75) $$1/k \exp(l_1 + \cdots + l_k) \leqq \varepsilon(\exp 2), \qquad k \geqq N.$$

Thus $\limsup (\exp(l_1 + \cdots + l_k))/k = 0$ and (2′) fails. Hence (2′)⇒(45). By Lemma 6, (1) also holds and so (2′)⇒(1).

On the other hand, (2′) is not necessary for covering as the following example shows.

EXAMPLE 1. $l = 1/n - \varepsilon/(n \log(n+1))$, $n = 1, 2, \cdots, 0 < \varepsilon \leqq 1$.

For this example, (1) holds and so covering takes place. However (2′) fails to hold.

Since (3) is known [1], [4] to be necessary for covering, we must have (1)⇒(3). A direct proof that (1)⇒(3) can be given as follows. By Lemma 6 (note that we may assume that l_n decreases to zero), if (1) holds,

$$
\begin{aligned}
\infty &= \sum_{n=1}^{\infty} (1/n - 1/(n+1)) \exp(l_1 + \cdots + l_n - nl_n) \\
&= \sum_{n=1}^{\infty} 1/n [\exp(l_1 + \cdots + l_{n+1} - (n+1)l_{n+1}) \\
&\qquad\qquad - \exp(l_1 + \cdots + l_n - nl_n)] \\
&= \sum_{n=1}^{\infty} 1/n [\exp(l_1 + \cdots + l_n)][\exp(-nl_{n+1})](1 - \exp(-n(l_n - l_{n+1})))
\end{aligned}
\tag{76}
$$

where we have summed by parts in the second line. Since $1 - \exp(-u) \leqq u$, we obtain, again summing by parts in the second line,

$$
\begin{aligned}
\infty &= \sum_{n=1}^{\infty} (l_n - l_{n+1}) \exp(l_1 + \cdots + l_n) \\
&= \sum_{n=2}^{\infty} l_n [\exp(l_1 + \cdots + l_n) - (l_1 + \cdots + l_{n-1})] \\
&= \sum_{n=2}^{\infty} l_n [\exp(l_1 + \cdots + l_n)](1 - \exp(-l_n)) \\
&\leqq \sum_{n=1}^{\infty} l_n^2 \exp(l_1 + \cdots + l_n).
\end{aligned}
\tag{77}
$$

Thus (1)⇒(3).

On the other hand, (3) is not sufficient for covering as the following example shows.

EXAMPLE 2. Define

$$
n(1) = 0, n(k) = 2^{2k} + n(k-1), \qquad k \geqq 2, \tag{78}
$$

$$\varepsilon(k) = (2^{k-1} - 1)/2^{2^k}, \qquad k \geqq 2 \tag{79}$$

$$l_j = \varepsilon(k)\log 2, \quad \text{for} \quad n(k-1) < j \leqq n(k). \tag{80}$$

Then for $k \geqq 2$,

$$l_1 + \cdots + l_{n(k)} = (2^k - k - 1)\log 2 \tag{81}$$

For $n(k-1) < n \leqq n(k)$,

$$l_1 + \cdots + l_n - nl_n = l_1 + \cdots + l_{n(k)} - n(k)l_{n(k)} \tag{82}$$

and so from (81) and (82),

$$\begin{aligned} &\sum_{n(k-1)<n\leqq n(k)} 1/n^2 \exp[l_1 + \cdots + l_n - nl_n] \\ &= \Bigg(\sum_{n(k-1)<n\leqq n(k)} 1/n^2\Bigg)\, 2^{2^k} - k - 1 - [(2^{k-1} - 1)n(k)/2^{2^k}] \end{aligned} \tag{83}$$

Since

$$n(k) \geqq 2^{2^k} \text{ and } \sum_{n(k-1)<n\leqq n(k)} 1/n^2 \leqq 1/(n(k-1)) < 1/2^{2^{k-1}} \tag{84}$$

we obtain from (83) by summing on k,

$$\begin{aligned} &\sum_{n=1}^{\infty} 1/n^2 \exp[l_1 + \cdots + l_n - nl_n] \\ &\leqq \sum_{k=2}^{\infty} 1/2^{2^{k-1}}\, 2^{2^{k-1}-k} < \infty. \end{aligned} \tag{85}$$

Thus (45) fails and by Lemma 6, (1) also fails. Thus Example 2 is not a case of covering. We show that nevertheless, (3) holds for Example 2.

We have from (80), summing a geometric progression,

$$\sum_{n(k-1)<n\leqq n(k)} l_n^2 \exp(l_1 + \cdots + l_n) \tag{86}$$

$$= (\varepsilon(k)\log 2)^2 \sum_{n(k-1)<n\leqq n(k)} \exp(l_1 + \cdots + l_{n(k-1)} + (n - n(k-1))l_{n(k)})$$

$$= (\varepsilon(k)\log 2)^2(1 - \exp(-l_{n(k)}))\,[\exp(l_1 + \cdots + l_{n(k)}) - \exp(l_1 + \cdots + l_{n(k-1)})]$$

Since by (80) and (81), as $k \to \infty$

$$(\varepsilon(k)\log 2)^2(1 - \exp(-l_{n(k)})) \sim \varepsilon(k)\log 2 \tag{87}$$

and the final square bracket in (86) is asymptotic as $k \to \infty$ to $\exp(l_1 + \cdots + l_{n(k)})$, summing (86) on k, we obtain

(88)
$$\sum_{n=1}^{\infty} l_n^2 \exp(l_1 + \cdots + l_n) \approx \sum_{k=1}^{\infty} \varepsilon(k) \exp(l_1 + \cdots + l_{n(k)})$$
$$\approx \sum_{k=1}^{\infty} (2^k / 2^{2^k})\, 2^{2^k - k} = \infty$$

where $\approx$ between series means the series both converge or both diverge. Thus (3) holds for Example 2 and so (3) is not sufficient for covering.

8. The case of equal lengths

There is no known simple formula for the probability of covering the circle with n arcs of *different* lengths. However, if $l_1 = l_2 = \cdots = l_n = \alpha$, Whitworth [9] showed that

(89)
$$P(U_n(\alpha) \not\supset C) = \sum_{1 \leqq k < 1/\alpha} (-1)^{k-1} \binom{n}{k} (1 - k\alpha)^{n-1}$$

where $U_n(\alpha)$ denotes the union of the n arcs of length α. For small α and large n the right side of (89) becomes difficult to estimate due to the violent oscillations of the summands. In such cases the methods of this paper provide convenient bounds for $P(U_n(\alpha) \not\supset C)$.

For $\alpha \leqq \frac{1}{4}$ an upper bound is obtained from (16) by setting $\varepsilon = \frac{1}{2}$, and using (34)

(90)
$$P(U_n(\alpha) \not\supset C) \leqq 4P(U_n(\alpha) \not\supset [0, \tfrac{1}{4}))$$
$$\leqq 2(1-\alpha)^{2n} \Big/ \left[\int_0^{\alpha} (1 - \alpha - t)^n \, dt + (\tfrac{1}{4} - \alpha)(1 - 2\alpha)^n \right],$$

while a lower bound is obtained by setting $\varepsilon = 1$ in (24), and noting that

$$Em^2(0,1) = 2 \int_0^{\frac{1}{2}} P(t \notin U_n, 0 \notin U_n) dt ,$$

(91)
$$P(U(\alpha) \not\supset C) \geqq \tfrac{1}{2}(1-\alpha)^{2n} \Big/ \left[\int_0^{\alpha} (1 - \alpha - t)^n dt + (\tfrac{1}{2} - \alpha)(1 - 2\alpha)^n \right].$$

We may use the above bounds to study the distribution of the random variable N_α, the first n for which $U(\alpha) \supset C$, for small values of α.

We note that

(92)
$$P(N_\alpha > n) = P(U(\alpha) \not\supset C).$$

Define

(93)
$$n(\alpha) = (1/\alpha) \log 1/\alpha + (1/\alpha) \log\log 1/\alpha.$$

We will show that as $\alpha \to 0$, $\alpha(N_\alpha - n(\alpha))$ has a proper limiting distribution (having exponential tails); in fact, for $-\infty < x < \infty$ fixed,

$$\frac{1}{2}(e^x + \tfrac{1}{2})^{-1} \leqq \lim_{\alpha \to 0} P(N_\alpha > n(\alpha) + x/\alpha) \leqq 2(e^x + \tfrac{1}{4})^{-1}. \tag{94}$$

To see this, set $n = n(\alpha) + x/\alpha$ in (90) and (91) and apply (92) and the following easily proved statement valid for any $a > 0$. If $n = n(\alpha) + x/\alpha$ with x fixed, then

$$\lim_{\alpha \to 0} (1-\alpha)^{2n} \Big/ \left[\int_0^\alpha (1-\alpha-t)^n dt + (a-\alpha)(1-2\alpha)^n \right] = (e^x + a)^{-1}. \tag{95}$$

In particular, from (94)

$$(N_\alpha - 1/\alpha \log 1/\alpha)/(1/\alpha \log\log 1/\alpha) \to 1 \tag{96}$$

in probability as $\alpha \to 0$.

Similarly, it follows from (90) and (91) that the expectation of N_α satisfies

$$EN_\alpha = n(\alpha) + 0(1/\alpha), \qquad \text{as } \alpha \to 0. \tag{97}$$

However, as far as (97) is concerned, Steutel [8] has obtained a sharper result (extending earlier work of Flatto and Konheim [3])

$$EN_\alpha = n(\alpha) + \gamma/\alpha + o(1/\alpha) \qquad \text{as } \alpha \to 0 \tag{98}$$

where γ is Euler's constant, by using Laplace transformation methods based on (89). Our inequalities are not sharp enough to obtain (98) or the limiting distribution of $\alpha(N_\alpha - n(\alpha))$. On the other hand it seems difficult to obtain the existence of a proper limiting distribution of $\alpha(N_\alpha - n(\alpha))$, or even (96), which is new, by methods based on (89) directly.

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